$(Y_1, Y_2):$ $(0,0)$ $(1,1)$ $(2,0)$ $(2,1)$ $(2,1)$ $(3,0)$

$P_{(Y_1,Y_2)}(Y_1, Y_2):$ $1/8$ $3/8$ $1/8$ $2/8$ $1/8$

$Y_1 = h_1(X_1, X_2, X_3) = X_1 + X_2 + X_3$

$Y_2 = h_2(X_1, X_2, X_3) = |X_3 - X_2|$

$(Y_1, Y_2) \sim ;$

Μάθημα 7ο

25/11/15

ΘΕΩΡΗΜΑ: Έστω $\underset{\sim}{X} = (X_1, \dots, X_n)$ συνεχές τ.δ. σ.π.π. $f_{\underset{\sim}{X}}(x)$ και $\underset{\sim}{Y} = (Y_1, \dots, Y_n) = (h_1(\underset{\sim}{X}), \dots, h_n(\underset{\sim}{X}))$ μετασχηματισμός. Θέτουμε $S = \{\underset{\sim}{x} : f_{\underset{\sim}{X}}(\underset{\sim}{x}) > 0\}$ και $T = \{\underset{\sim}{y} : y = (y_1, \dots, y_n), y_i = h_i(x), i = 1, \dots, n, \underset{\sim}{x} \in S\}$.

Υποθέτουμε ότι ① Υπάρχει μια διαμέριση $S_1, \dots, S_k$ του S (τα S_j ξένα μεταξύ τους τέτοια ώστε για κάθε S_j η συνάρτηση $\underset{\sim}{y} = (y_1 = h_1(\underset{\sim}{x}), \dots, y_n = h_n(\underset{\sim}{x}))$ είναι αμφιμονοσήμαντη απεικόνιση του S_j στο T.

Έστω $y_1 = h_{1j}(\underset{\sim}{x}), \dots, y_n = h_{nj}(\underset{\sim}{x})$ ο περιορισμός των $h_1, \dots, h_n$ στο S_j και $T_j = \{\underset{\sim}{y} : y_1 = h_{1j}(\underset{\sim}{x}), \dots, y_n = h_{nj}(\underset{\sim}{x}), \underset{\sim}{x} \in S\}$, $j = 1, \dots, k$ και $T = \bigcup_{j=1}^{k} T_j$ αλλά τα T_j δεν είναι κατ' ανάγκη ξένα μεταξύ τους.

② Για κάθε j οι αντίστροφες συναρτήσεις $h_{1j}^{-1}(\underset{\sim}{y}), \dots, h_{nj}^{-1}(\underset{\sim}{y})$ έχουν όλες τις μερικές παραγώγους πρώτης τάξης και οι παράγωγοι αυτοί είναι συνεχείς για κάθε $\underset{\sim}{y} \in T_j$

Για κάθε j, ορίζουμε την Ιακωβιανή ορίζουσα J_j

$$\left| \left(\frac{dh_{ij}^{-1}(\underset{\sim}{y})}{dy_l} \right) \right| = J_j = \begin{vmatrix} \frac{dh_{1j}^{-1}(\underset{\sim}{y})}{dy_1} & \frac{dh_{1j}^{-1}(\underset{\sim}{y})}{dy_2} & \cdots & \frac{dh_{1j}^{-1}(\underset{\sim}{y})}{dy_n} \\ \vdots & \vdots & & \vdots \\ \frac{dh_{nj}^{-1}(\underset{\sim}{y})}{dy_1} & \frac{dh_{nj}^{-1}(\underset{\sim}{y})}{dy_2} & \cdots & \frac{dh_{nj}^{-1}(\underset{\sim}{y})}{dy_n} \end{vmatrix} \ (\neq 0).$$

$i = 1, \dots, n$
$l = 1, \dots, n$
$j = 1, \dots, k$

Τότε, η σ.π.π. της $\underset{\sim}{Y} = (Y_1, \dots, Y_n)$ δίνεται από τη σχέση:

$f_{\underset{\sim}{Y}}(\underset{\sim}{y}) = \sum_j f_{\underset{\sim}{X}}(h_{1j}^{-1}(\underset{\sim}{y}), \dots, h_{nj}^{-1}(\underset{\sim}{y}))\,|J_j|\,, \ \underset{\sim}{y} \in T$ όπου το άθροισμα είναι για

εκείνα τα j για κάποια $h_{1j}(\underset{\sim}{x}) = y_1, \dots, h_{nj}(\underset{\sim}{x}) = y_n$
(για τα κοινά T_j)

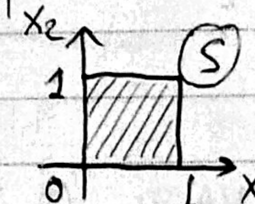
$f_{\underset{\sim}{Y}}(\underset{\sim}{y}) = f_{\underset{\sim}{X}}(h_1^{-1}(\underset{\sim}{y}), \dots, h_n^{-1}(\underset{\sim}{y}))\,|J|$

↑ με αντικατάσταση $f_{\underset{\sim}{X}}(\underset{\sim}{x})$

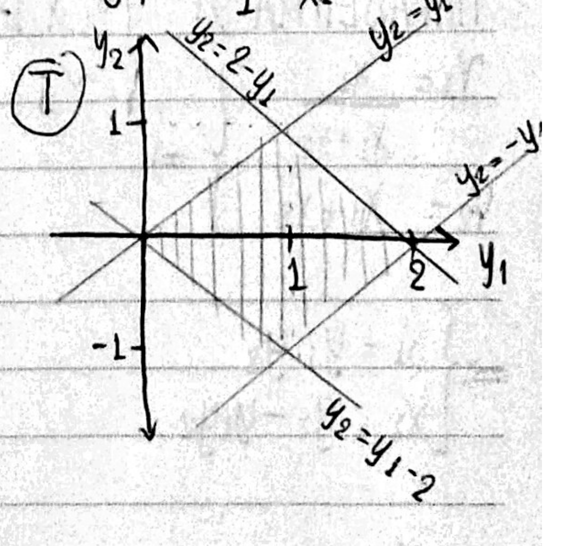
$$J = \left|\left(\frac{dx_i}{dy_j}\right)\right| \quad , i = 1, \dots, n \quad j = 1, \dots, n.$$

ΠΑΡΑΔΕΙΓΜΑ 1: $(X_1, X_2) \sim f_{X_1,X_2}(x_1,x_2) = 1, \ 0 < x_1 < x_2 < 1$ || $f_{Y_1,Y_2}(y_1,y_2) = ;, \ f_{Y_1}(y_1), f_{Y_2}(y_2)$

$$\left.\begin{aligned} Y_1 &= X_1 + X_2 \\ Y_2 &= X_2 - X_1 \end{aligned}\right\} \Rightarrow$$



$$\Rightarrow \begin{cases} x_1 = \dfrac{y_1 - y_2}{2} \\ x_2 = \dfrac{y_1 + y_2}{2} \end{cases} \quad \begin{cases} 0 < y_1 < 2 \\ -1 < y_2 < 1 \end{cases} \ \text{και}$$



και $\quad 0 < y_1 - y_2 < 2 \ \begin{cases} y_2 < y_1 \\ y_2 > y_1 - 2 \end{cases}$

$\quad 0 < y_1 + y_2 < 2 \ \begin{cases} y_2 > -y_1 \\ y_2 < 2 - y_1 \end{cases}$

$$J = \begin{vmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{vmatrix} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2} \neq 0.$$

$f_{Y_1,Y_2}(y_1,y_2) = \frac{1}{2}, \quad (y_1,y_2) \in T$

$f_{Y_1}(y_1) = \int \frac{1}{2}\, dy_2 \quad , 0 < y_1 < 2$

$$f_{Y_1}(y_1) = \int_{-y_1}^{y_1} \frac{1}{2}\, dy_2 = y_1 \quad , 0<y_1<1$$

$$= \int_{y_1-2}^{2-y_1} \frac{1}{2}\, dy_2 = 2-y_1 \quad , 1<y_1<2$$

$$\boxed{\int_0^1 y_1\, dy_1 + \int_1^2 (2-y_1)\, dy_1 = 1.}$$

$$f_{Y_2}(y_2) = \int \frac{1}{2}\, dy_1 = \quad , -1<y_2<1$$

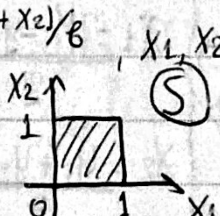
$$= \int_{y_2}^{y_2+2} \frac{1}{2}\, dy_1 = 1+y_2 \quad , -1<y_2<0$$

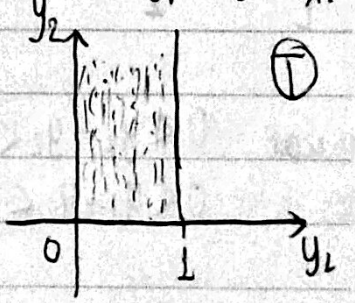
$$\boxed{\int_{-1}^{0} (1+y_2)\, dy_2 + \int_0^1 (1-y_2)\, dy_2 = 1}$$

$$= \int_{y_2}^{2-y_2} \frac{1}{2}\, dy_1 = 1-y_2 \quad , 0<y_2<$$

ΠΑΡΑΔΕΙΓΜΑ 2: $(X_1, X_2) \sim f_{X_1,X_2}(x_1,x_2) = \dfrac{x_1^{a_1-1}\, x_2^{a_2-1}}{\beta^{a_1+a_2}\,\Gamma(a_1)\Gamma(a_2)}\, e^{-(x_1+x_2)/\beta}$, $x_1, x_2 > 0$.

$$\left.\begin{aligned} Y_1 &= \frac{X_1}{X_1+X_2} \\ Y_2 &= X_1+X_2 \end{aligned}\right\} \Rightarrow \quad , Y_1 \sim ;$$





$$\Rightarrow \begin{cases} x_1 = y_1 y_2 \\ x_2 = y_2 - y_1 y_2 \end{cases}$$

$$J = \begin{vmatrix} y_2 & y_1 \\ -y_2 & 1-y_1 \end{vmatrix} = y_2(1-y_1) + y_1 y_2 = y_2 \neq 0.$$

$$f_{Y_1,Y_2}(y_1,y_2) = \frac{1}{\beta^{a_1+a_2}\,\Gamma(a_1)\Gamma(a_2)} (y_1 y_2)^{a_1-1} (y_2 - y_1 y_2)^{a_2-1} e^{-y_2/\beta}$$

$$= \frac{1}{\beta^{a_1+a_2}\,\Gamma(a_1)\Gamma(a_2)}\, y_1^{a_1-1} (1-y_1)^{a_2-1}\, y_2^{a_1+a_2-1} e^{-y_2/\beta} \quad , y_2>0, \; 0<y_1<1.$$

$$f_{Y_1}(y_1) = \int_0^\infty f_{Y_1,Y_2}(y_1,y_2)\,dy_2 = \frac{1}{\beta^{a_1+a_2}\Gamma(a_1)\Gamma(a_2)}\, y_1^{a_1-1}(1-y_1)^{a_2-1}\underbrace{\int_0^\infty y_2^{a_1+a_2-1}e^{-y_2/\beta}\,dy_2}_{\parallel}$$

$$= \frac{1}{\cancel{\beta^{a_1+a_2}}\,\Gamma(a_1)\Gamma(a_2)}\, y_1^{a_1-1}(1-y_1)^{a_2-1}\,\cancel{\beta^{a_1+a_2}}\,\Gamma(a_1+a_2) =$$

$$= \frac{\Gamma(a_1+a_2)}{\Gamma(a_1)\Gamma(a_2)}\, y_1^{a_1-1}(1-y_1)^{a_2-1}, \quad 0<y_1<1.$$

δηλ. $Y_1 \equiv Be(a_1,a_2)$.

$$\boxed{\frac{1}{B(a_1,a_2)} = \frac{\Gamma(a_1+a_2)}{\Gamma(a_1)\Gamma(a_2)}}$$

$$f_{Y_2}(y_2) = \frac{1}{\beta^{a_1+a_2}\Gamma(a_1)\Gamma(a_2)}\, y_2^{a_1+a_2-1}e^{-y_2/\beta}\underbrace{\int_0^1 y_1^{a_1-1}(1-y_1)^{a_2-1}\,dy_1}_{\parallel}$$

$$= \frac{1}{\beta^{a_1+a_2}\,\cancel{\Gamma(a_1)\Gamma(a_2)}}\, y_2^{a_1+a_2-1}e^{-y_2/\beta}\,\frac{\cancel{\Gamma(a_1)\Gamma(a_2)}}{\Gamma(a_1+a_2)} =$$

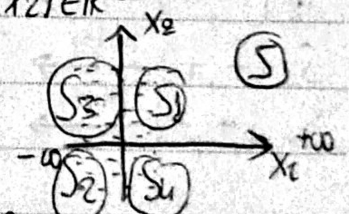
$$= \frac{y_2^{a_1+a_2-1}e^{-y_2/\beta}}{\beta^{a_1+a_2}\Gamma(a_1+a_2)}, \quad y_2>0.$$

$$\equiv G(a_1+a_2,\beta)$$

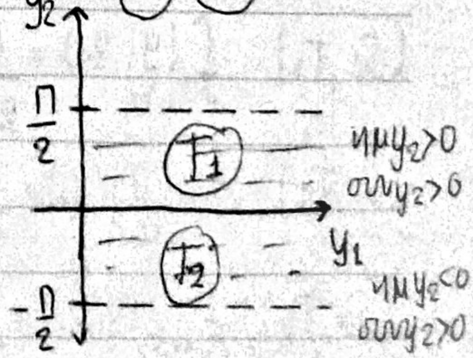
ΠΑΡΑΔΕΙΓΜΑ 3: $(X_1,X_2) \sim f_{X_1,X_2}(x_1,x_2) = \frac{1}{2\pi} e^{-(x_1^2+x_2^2)/2}, \quad (x_1,x_2)\in\mathbb{R}^2$

$Y_1 = (X_1^2+X_2^2)^{1/2}, \quad y_1>0$

$Y_2 = \text{τοξεφ}\left(\frac{X_2}{X_1}\right), \quad -\frac{\pi}{2}<y_2<\frac{\pi}{2}$



$f_{Y_1,Y_2}(y_1,y_2),\ f_{Y_1}(y_1),\ f_{Y_2}(y)$



$$\left.\begin{aligned} X_1^2+X_2^2 &= Y_1^2 \\ \frac{X_2}{X_1} &= \text{εφ}\,Y_2 \end{aligned}\right\} \Rightarrow \begin{aligned} x_1^2+x_2^2 &= y_1^2 \\ x_2 &= x_1\,\text{εφ}\,y_2 \end{aligned} \Rightarrow x_1^2 + x_1^2\,\text{εφ}^2 y_2 = y_1^2$$

$$\Rightarrow x_1^2\left(1+\frac{\text{ημ}^2 y_2}{\text{συν}^2 y_2}\right) = y_1^2 \Rightarrow x_1^2(\text{ημ}^2 y_2 + \text{συν}^2 y_2) = y_1^2\,\text{συν}^2 y_2 \Rightarrow$$

$x_1^2 = y_1^2 συν^2 y_2 \Rightarrow x_1 = \pm y_1 συν y_2$.

$x_1 = y_1 συν y_2$

$x_2 = y_1 \cancel{συν y_2} \cdot \dfrac{ημ y_2}{\cancel{συν y_2}} = y_1 ημ y_2$

$x_1 = y_1 συν y_2$ $\quad M(+)$

$x_2 = y_1 ημ y_2$

$x_1 = -y_1 συν y_2$ $\quad M(-)$

$x_2 = -y_1 ημ y_2$

$$M(+): \quad J = \begin{vmatrix} συν y_2 & -y_1 ημ y_2 \\ ημ y_2 & y_1 συν y_2 \end{vmatrix} = y_1 συν^2 y_2 + y_1 ημ^2 y_2 = y_1 \neq 0$$

$$M(-): \quad J = \begin{vmatrix} -συν y_2 & y_1 ημ y_2 \\ -ημ y_2 & -y_1 συν y_2 \end{vmatrix} = y_1 συν^2 y_2 + y_1 ημ^2 y_2 = y_1 \neq 0$$

$S_1(+,+) \xrightarrow{M(+)} T_1(+,+)$

$S_2(-,-) \xrightarrow{M(-)} T_1(+,+)$

$S_3(-,+) \xrightarrow{M(-)} T_2(+,-)$

$S_4(+,-) \xrightarrow{M(+)} T_2(+,-)$

$$\left.\begin{aligned} (S_1,T_1): \; f_{Y_1,Y_2}(y_1,y_2) &= \frac{y_1}{2\pi} e^{-y_1^2/2}, \quad (y_1,y_2) \in T_1 \\ (S_2,T_1): \; f_{Y_1,Y_2}(y_1,y_2) &= \frac{y_1}{2\pi} e^{-y_1^2/2}, \quad (y_1,y_2) \in T_1 \end{aligned}\right\} \Rightarrow ①$$

$$\left.\begin{aligned} (S_3,T_2): \; f_{Y_1,Y_2}(y_1,y_2) &= \frac{y_1}{2\pi} e^{-y_1^2/2}, \quad (y_1,y_2) \in T_2 \\ (S_4,T_2): \; f_{Y_1,Y_2}(y_1,y_2) &= \frac{y_1}{2\pi} e^{-y_1^2/2}, \quad (y_1,y_2) \in T_2 \end{aligned}\right\} \Rightarrow ②$$

$$① \Rightarrow f_{Y_1,Y_2}(y_1,y_2) = \frac{1}{\pi} y_1 e^{-y_1^2/2}, \quad (y_1,y_2) \in T_1$$

$$② \Rightarrow f_{Y_1,Y_2}(y_1,y_2) = \frac{1}{\pi} y_1 e^{-y_1^2/2}, \quad (y_1,y_2) \in T_2$$

$$\Rightarrow f_{Y_1,Y_2}(y_1,y_2) = \frac{1}{\pi} y_1 e^{-y_1^2/2}, \quad (y_1,y_2) \in T$$

$$f_{Y_1}(y_1) = y_1 e^{-y_1^2/2} \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dy_2 = y_1 \cdot e^{-y_1^2/2}, \quad y_1 > 0.$$

$$f_{Y_2}(y_2) = \frac{1}{\pi} \int_0^{\infty} y_1 e^{-y_1^2/2} dy_1 = \frac{1}{\pi} \int_0^{\infty} e^{-y_1^2/2} d(y_1^2/2) = \frac{1}{\pi}\left[-e^{-y_1^2/2}\right]_0^{\infty} = \frac{1}{\pi}\left[-\frac{1}{e^{\infty}} + \frac{1}{e^{0}}\right] = \frac{1}{\pi}$$

$$-\frac{\pi}{2} < y_2 < \frac{\pi}{2}$$

Κατανομή αθροισμάτων τ.μ. - Συνελίξεις

1. Έστω X και Y διακριτές τ.μ. με σ.π. $P_{X,Y}(x,y)$ και $Z = X+Y$ και $W = X-Y$

- $P_Z(z) = P(Z=z) = \sum_y P_{X,Y}(z-y,y) = \sum_x P_{X,Y}(x, z-x)$

$P_W(w) = P(W=w) = \sum_y P_{X,Y}(w+y,y) = \sum_x P_{X,Y}(x, x-w)$.

ΑΠΟΔΕΙΞΗ:

- $P_Z(z) = \sum_y P_{Z,Y}(z,y) = \sum_y P(Z=z, Y=y) = \sum_y P(Z=z \mid Y=y) P(Y=y) =$

$= \sum_y P(X+Y=z \mid Y=y) P(Y=y) = \sum_y P(X=z-y \mid Y=y) P(Y=y) =$

$= \sum_y P(X=z-y, Y=y) = \sum_y P_{X,Y}(z-y,y)$.

2. Έστω X και Y συνεχείς τ.μ. με σ.π.π. $f_{X,Y}(x,y)$ και $Z = X+Y$ και $W = X-Y$

- $f_Z(z) = \int_{-\infty}^{+\infty} f_{X,Y}(z-y,y)\,dy = \int_{-\infty}^{+\infty} f_{X,Y}(x, z-x)\,dx$

$f_W(w) = \int_{-\infty}^{+\infty} f_{X,Y}(w+y,y)\,dy = \int_{-\infty}^{+\infty} f_{X,Y}(x, x-w)\,dx$.

ΑΠΟΔΕΙΞΗ:

$$\left.\begin{array}{l} Z = X+Y \\ Y = y \end{array}\right\} \Rightarrow \begin{array}{l} X = t-y \\ y = y \end{array} \Rightarrow J = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1 \rightarrow$$

$$f_{Z,Y}(z,y) = f_{X,Y}(z-y,y)\,|1| \Rightarrow f_Z(z) = \int_{-\infty}^{+\infty} f_{X,Y}(z-y,y)\,dy$$

Για X, Y ανεξάρτητες.

$P_{X,Y}(x,y) = P_X(x) \cdot P_Y(y)$

$$P_Z(z) = \sum_y P_X(z-y)\,P_Y(y) \leftarrow P_Z(z) = P_X(x) * P_Y(y)$$

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(z-y) f_Y(y)\,dy \leftarrow f_Z(z) = f_X(x) * f_Y(y)$$